PSEUDOGRAPH AND ITS ASSOCIATED REAL TORIC MANIFOLD

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ABSTRACT. Given a simple graph G, the graph associahedron P_G is a convex polytope whose facets correspond to the connected induced subgraphs of G. Graph associahedra have been studied widely and are found in a broad range of subjects. Recently, S. Choi and H. Park computed the rational Betti numbers of the real toric variety corresponding to a graph associahedron under the canonical Delzant realization. In this paper, we focus on a pseudograph associahedron which was introduced by Carr, Devadoss and Forcey, and then discuss how to compute the Poincaré polynomial of the real toric variety corresponding to a pseudograph associahedron under the canonical Delzant realization.

1. INTRODUCTION

A *toric variety* of complex dimension n is a normal algebraic variety over $\mathbb{C}$ with an effective algebraic action of $(\mathbb{C} \setminus \{O\})^n$ having an open dense orbit. A compact smooth toric variety is called a *toric manifold*, and the subset consisting of points with real coordinates of a toric manifold is called a *real toric manifold*. One of the most important facts in toric geometry, the so called *fundamental theorem of toric geometry*, is that there is a 1-1 correspondence between the class of toric varieties of complex dimension n and the class of fans in $\mathbb{R}^n$. In particular, a toric manifold X of complex dimension n corresponds to a complete non-singular fan Σ_X in $\mathbb{R}^n$. Furthermore, if X is projective, then Σ_X can be realized as the normal fan of some simple polytope of real dimension n. A simple polytope P of dimension n is called *Delzant* if for each vertex $p \in P$, the outward normal vectors of the facets containing p can be chosen to make up an integral basis for $\mathbb{Z}^n$. Note that the normal fan of a Delzant polytope is a complete non-singular fan and thus defines a projective toric manifold and a real toric manifold by the fundamental theorem as well.

The (integral) Betti numbers of a toric manifold X of complex dimension n are easily computed. It is known by the theorem of Danilov[7]-Jurkiewicz[8] that the Betti

numbers of X vanish in odd degrees and the $2i$th Betti number of X is equal to h_i, where $(h_0, h_1, \ldots, h_n)$ is the h-vector of Σ_X. Unlike toric manifolds, however, only little is known about the topology of real toric manifolds. In [10] and [11], Suciu and Trevisan have found a formula for the rational cohomology groups of a real toric manifold, see also [6]. Let P be a Delzant polytope of dimension n. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be the set of facets of P. Then, the outward normal vectors of P can be understood as a function ϕ from $\mathcal{F}$ to $\mathbb{Z}^n$, and the composition map $\lambda \colon \mathcal{F} \xrightarrow{\phi} \mathbb{Z}^n \xrightarrow{\mathrm{mod}\,2} \mathbb{Z}_2^n$ is called the (mod 2) *characteristic function* over P. Note that λ can be represented by a $\mathbb{Z}_2$-matrix Λ of size $n \times m$ as

$$\Lambda = \begin{pmatrix} \lambda(F_1) & \cdots & \lambda(F_m) \end{pmatrix},$$

where the ith column of Λ is $\lambda(F_i) \in \mathbb{Z}_2^n$. For $\omega \in \mathbb{Z}_2^m$, we define P_ω to be the union of facets F_j such that the jth entry of ω is nonzero. Denote by $M_\lambda(P)$ the real toric manifold corresponding to P and λ. Then the ith rational Betti number of $M_\lambda(P)$ is given by

$$(1.1) \qquad \beta^i(M_\lambda(P)) = \sum_{\omega \in \mathrm{Row}(\lambda)} \mathrm{rank}_{\mathbb{Q}}\, \widetilde{H}^{i-1}(P_\omega; \mathbb{Q}),$$

where $\mathrm{Row}(\lambda)$ is the space of m-dimensional $\mathbb{Z}_2$-vectors spanned by the rows of Λ associated with λ. S. Choi and H. Park [6] reproved the formula (1.1) by using different methods, and showed that it holds even for the cohomology group of arbitrary coefficient ring in which 2 is a unit. Furthermore, it is known in [4] that the suspension of M localized at an odd prime p (or $p = 0$, which is known as the rationalization) can be decomposed as

$$(1.2) \qquad \Sigma M \simeq_p \Sigma \bigvee_{\omega \in \mathrm{Row}(\lambda)} \Sigma P_\omega.$$

Recently, the rational Betti numbers of some interesting family of real toric manifolds, arising from simple graphs, have been computed by using (1.1) in [5]. Let G be a simple graph with node set $[n+1] := \{1, \ldots, n+1\}$ and $\mathcal{B}(G)$ the *graphical building set* which is the collection of subsets of $[n+1]$ all of whose elements are obtained from connected induced subgraphs of G. For $I \subset [n+1]$, let Δ_I be the simplex given by the convex hull of points $\mathbf{e}_i$, $i \in I$, where $\mathbf{e}_i$ is the ith standard basis vector. Then define the *graph associahedron* P_G as the Minkowski sum of simplices

$$(1.3) \qquad P_G = \sum_{I \in \mathcal{B}(G)} \Delta_I.$$

One can see that G is connected if and only if P_G is n-dimensional. We note that, for every connected graph G with $n+1$ nodes, there is a canonical way to realize P_G as a Delzant polytope from Δ^n in $\mathbb{R}^n$ by truncating its faces corresponding to elements of $\mathcal{B}(G) \setminus \{[n+1]\}$ (refer [12] or [9, Proposition 7.10]). We note that to each facet of P_G there is an assigned element I of $\mathcal{B}(G) \setminus \{[n+1]\}$. In this case, by choosing an

appropriate basis of $\mathbb{Z}_2^n$, the characteristic function λ_G is

$$\lambda_G(I) = \begin{cases} \sum_{i \in I} \mathbf{e}_i, & \text{if } n+1 \notin I; \\ \sum_{i \notin I} \mathbf{e}_i, & \text{if } n+1 \in I. \end{cases}$$

We denote by M_G the real toric manifold $M_{\lambda_G}(P_G)$ corresponding to P_G under the canonical Delzant realization.

In [5], the rational Betti numbers of M_G are computed in terms of the a-numbers of G. The authors noted that, for each simple graph G with an even number of nodes, there is the unique element $\omega_0(G) \in \mathrm{Row}(\lambda_G)$ such that the component of $\omega_0(G)$ corresponding to every singleton in $\mathcal{B}(G)$ is nonzero. The remarkable fact discovered in [5] is that for $\omega \in \mathrm{Row}(\lambda_G)$, $(P_G)_\omega$ is homotopy equivalent to $(P_H)_{\omega_0(H)}$ for some subgraph H of G induced by an even number of nodes. Furthermore, $(P_H)_{\omega_0(H)}$ is homotopy equivalent to a wedge of spheres of the same dimension. Therefore, they concluded that $(P_G)_\omega$ is homotopy equivalent to a wedge of spheres of the same dimension for each $\omega \in \mathrm{Row}(\lambda_G)$.

In this paper, we discuss the pseudographical analogue of the above phenomena. A *pseudograph* G is a graph in which both loops and multiple edges are permitted. The notion of a pseudograph associahedron is originally introduced in [3] as a generalization of a graph associahedron. Indeed, it coincides with the graph associahedron P_G if G is a connected simple graph. However, if G is disconnected, then it does not coincide with P_G since the dimension of the corresponding pseudograph associahedron in [3] is always greater than the dimension of the graph associahedron. So in this paper, we introduce a slightly modified definition of a pseudograph associahedron, see Section 2, which totally agrees with the notion of a graph associahedron when G is simple. Then, we will denote again by P_G the (our modified) pseudograph associahedron of a pseudograph G. We also remark that if G does not have a loop, then there is the canonical Delzant realization of P_G, and, hence, it provides a real toric manifold M_G.

For a topological space X, if X has the homotopy type of a finite CW-complex, then we let $\mathrm{Poin}_X(t)$ and $\widetilde{\mathrm{Poin}}_X(t)$ be the polynomials defined by

$$\mathrm{Poin}_X(t) = \sum_{i=0}^{\infty} \beta^i(X)t^i \quad \text{and} \quad \widetilde{\mathrm{Poin}}_X(t) = \sum_{i=0}^{\infty} \tilde{\beta}^i(X)t^i,$$

where $\beta^i(X)$ is the ith rational Betti number of X and $\tilde{\beta}^i(X)$ is the ith reduced rational Betti number of X. The polynomial $\mathrm{Poin}_X(t)$ is called the *Poincaré polynomial* of X. Our main result is about computing the Poincaré polynomial of M_G.

A pseudograph H is an *induced subgraph* of G if H is a subgraph that includes all edges between every pair of nodes in H if such edges of G exist. A pseudograph G' is a *partial underlying pseudograph* of G if G' can be obtained from G by replacing some bundles with simple edges. We denote by $H \lessdot G$ if H is a partial underlying pseudograph of an induced subgraph of G. The main result of this paper may be stated as follows, and all definitions and notations related to Theorem 1.1 are explained in Section 3, with more observations and examples. Throughout this paper, for simplicity,

we also denote by K the topological realization of a simplicial complex K if there is no danger of confusion.

Theorem 1.1. *For any pseudograph G, we have*

$$\mathrm{Poin}_{M_G}(t) = 1 + t \sum_{H \lessdot G} \sum_{\substack{C \subset \mathcal{C}_H \\ admissible\ to\ H}} \widetilde{\mathrm{Poin}}_{K_{C,H}^{\mathrm{odd}}}(t),$$

where $\mathcal{C}_H$ is the set of all nodes and all multiple edges of H.

Roughly, for $C \subset \mathcal{C}_H$, C is said to be admissible to H if C has an even number of nodes and an even number of multiple edges of H satisfying certain conditions, and $K_{C,H}^{\mathrm{odd}}$ is a subcomplex of the dual of ∂P_H satisfying certain conditions related to C.

In our formula, $\sum_{\substack{C \subset \mathcal{C}_H \\ admissible\ to\ H}} \widetilde{\mathrm{Poin}}_{K_{C,H}^{\mathrm{odd}}}(t)$ is completely determined by $H \lessdot G$, and so we define it by the *a-polynomial* $a_H(t)$ of H. Interestingly, our main theorem says that for the computation of $\mathrm{Poin}_{M_G}(t)$, it is sufficient to consider some subgraphs of G, instead of considering P_ω for all $\omega \in \mathrm{Row}(\lambda_G)$. One of the main ideas to prove Theorem 1.1 is to show that for $\omega \in \mathrm{Row}(\lambda_G)$, $(P_G)_\omega$ is homotopy equivalent to $K_{C,H}^{\mathrm{odd}}$ for some collection C and some $H \lessdot G$, see Proposition 4.7. Hence, not only this result reduces the computation of $\mathrm{Poin}_{M_G}(t)$ remarkably, but enhances the understanding of the topology of suspended M_G as (1.2), since the dimension of $K_{C,H}^{\mathrm{odd}}$ is much less than that of $(P_G)_\omega$.

Our result is a generalization of [5] in a sense that the *a-polynomial* of a simple graph H corresponds to the *a-number* of H. More precisely, if a graph G is simple, then an induced subgraph H of G having an admissible collection must have an even number of nodes and its node set is a unique admissible collection of H, and therefore the *a-polynomial* $a_H(t)$ is a monomial of degree $\frac{|V(H)|}{2}$ and the coefficient of the monomial is exactly equal to the *a-number* of H.

This paper is organized as follows: in Section 2, we review the slightly modified definition of a pseudograph associahedron, and show that pseudograph associahedra are Delzant. In Section 3, we define the *a-polynomial* of a pseudograph, and prepare the notions necessary to state the main theorem. In Section 4, we show that $(P_G)_\omega$ is homotopy equivalent to $K_{C,H}^{\mathrm{odd}}$ and prove the main theorem. We conclude in Section 5 with some remarks.

2. Pseudograph associahedron and its associated real toric manifold

In this section, we briefly review the construction of the pseudograph associahedron P_G for a pseudograph G based on [3]. However, we modify some notions in order to improve the explanation, but the essential ideas are the same.

A *pseudograph* G is an ordered pair $G := (V, E)$, where V is a set of *nodes*[1] and E is a multiset of unordered pairs of nodes, called *edges*. If the endpoints of an edge e are

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[1]In this paper, we use 'node' for a graph or a pseudograph, and we use 'vertex' for a 0-dimensional simplex.

the same then e is called a *loop*. A pseudograph is said to be *finite* if both V and E are finite sets. Throughout this paper, we only consider a finite pseudograph. An edge $e \in E$ is said to be *multiple* if there exists an edge e' ($\neq e$) in E such that e and e' have the same pair of endpoints. An edge is called a *simple edge* if it is not a multiple edge. A *bundle* is a maximal set of multiple edges which have the same pair of endpoints. For example, the pseudograph G in Figure 1 on page 6 has five multiple edges and two bundles.

The *underlying simple graph* of G is created by deleting all loops and replacing each bundle with a simple edge. A pseudograph G is said to be *connected* if its underlying simple graph is connected. A pseudograph H is an *induced* (respectively, *semi-induced*) subgraph of G if H is a subgraph that includes all edges (respectively, at least one edge) between every pair of nodes in H if such edges exist in G. Note that for a simple graph, semi-induced subgraphs and induced subgraphs are the same concept. For example, in Figure 1, I_1, I_4, and I_5 are induced subgraphs and I_3 is a semi-induced subgraph of G.

Definition 2.1. Let G be a (not necessarily connected) pseudograph. A *tube* is a proper connected semi-induced subgraph I of some connected component of G.[2] A tube is said to be *full* if it is an induced subgraph of G. Two tubes *meet by inclusion* if one properly contains the other, and they *meet by separation* if they are disjoint and cannot be connected by an edge of G. Two tubes are *compatible* if they meet by inclusion or separation. A *tubing* of G is a set of pairwise compatible tubes.

From now on, we will consider only pseudographs without loops. We give labels to the nodes and the multiple edges of a pseudograph. In addition, when we consider a subgraph H of a pseudograph G, the labels of H are inherited from the labels of G. Thus, if a pseudograph H is considered as a subgraph of a pseudograph G, then H might have a labelled simple edge, which is not in a bundle of H (actually, it is in a bundle of G). Here are examples for tubes and tubings.

Example 2.2. Let G be a pseudograph labelled as Figure 1. Then G has two bundles $\{a, b\}$ and $\{c, d, e\}$. The subgraphs I_1, I_3, I_4, and I_5 are tubes of G, but I_2 is not a tube. In particular, I_1, I_4, and I_5 are full tubes. Note that $\{I_1, I_4, I_5\}$ is a tubing since I_1 and I_4 meet by inclusion, I_4 and I_5 meet by inclusion, and I_1 and I_5 meet by separation. But, $\{I_1, I_3, I_4\}$ is not a tubing since I_3 and I_4 are not compatible.

For a pseudograph G, a subgraph I of G will be denoted by the set of nodes of I and edges of I in a bundle of G. For instance, for the five subgraphs of G in Figure 1,

$$I_1 = \{2\}, \quad I_2 = \{1, 2, 4\}, \quad I_3 = \{1, 2, a\}, \quad I_4 = \{2, 4, 5, c, d, e\}, \quad \text{and} \quad I_5 = \{5\}.$$

It should be noted that for a semi-induced subgraph I, this set expression makes sense because I is the subgraph of G induced by the corresponding set. In the same sense, for a subgraph I of G, we denote by $\alpha \in I$ if α is a node of I or a multiple edge of G.

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[2]This is different from the definition of a tube in [2] by Carr and Devados. They defined a tube by a proper connected subgraph that includes at least one edge between every pair of nodes in I if such edges exist, and so a connected component itself can be a tube while our definition does not allow it.
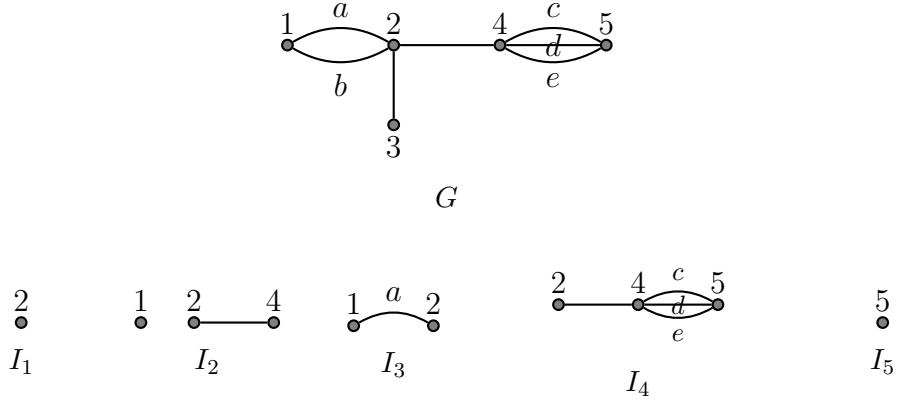
FIGURE 1. A pseudograph G and its subgraphs

From now on, for simplicity, we omit the braces and commas, and we always denote it in a way that the nodes proceed to the multiple edges, and the nodes are arranged in increasing order, like as $I_1 = 2$, $I_3 = 12a$, $I_4 = 245cde$, and $I_5 = 5$.

Before we define a polyhedron associated with a pseudograph G, we introduce a specific labelling L_I corresponding to a tube I of G. For each tube I of G, L_I is defined to be the minimal collection of nodes and edges of G such that

- L_I contains I as a set, and
- if a bundle B of G satisfies that $B \cap I = \emptyset$, then $B \subset L_I$.

For simplicity, we also omit the braces and commas when we denote L_I. For instance, the tubes I_1, I_3, I_4, and I_5 in Figure 1 have the associated labels

$$L_{I_1} = 2abcde, \quad L_{I_3} = 12acde, \quad L_{I_4} = 245abcde, \quad and \quad L_{I_5} = 5abcde.$$

Now we are ready to define a pseudograph associahedron corresponding to a finite pseudograph without loops.[3] Let G be a connected pseudograph with $n+1$ nodes. Let $B_1, \ldots, B_k$ be the bundles of G, and $b_i + 1$ the number of edges in B_i for $i = 1, \ldots, k$. Define Δ_G as the product

$$\Delta_G := \Delta^n \times \prod_{i=1}^{k} \Delta^{b_i}$$

of simplices endowed with the following labels on its faces:

(1) each facet of the simplex Δ^n is labelled with a particular node of G, and each face of Δ^n corresponds to a proper subset of nodes of G, defined by the intersection of the facets associated with those nodes;

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[3]In [3], a pseudograph associahedron is defined for any pseudograph even if it has a loop. However, in this case, the pseudograph associahedron is not bounded. In this paper, since we are interested in only Delzant polytopes that are bounded, we shall deal with only pseudographs having no loops.

(2) each vertex of the simplex Δ^{b_i} is labelled with a particular edge in B_i, and each face of Δ^{b_i} corresponds to a subset of B_i defined by the vertices spanning the face; and

(3) these labels naturally induce a labelling on Δ_G.

We note that a full tube of G is determined by the element of the graphical building set $\mathcal{B}(G^s)$, where G^s is the underlying simple graph of G. Hence, one can see that, for a connected pseudograph G, truncating the faces of Δ_G labelled by L_I for full tubes I, in increasing order of dimension, constructs

$$\Delta_G^* := P_{G^s} \times \prod_{i=1}^{k} \Delta^{b_i}.$$

As each face f of Δ_G^* corresponding to a full tube is truncated, those subfaces of f that correspond to tubes but have not yet been truncated are removed. It is natural, however, to assign these defunct tubes to the combinatorial images of their original subfaces.

Definition 2.3 (Modifying the definition in [3]). Let G be a pseudograph. If G is connected, then truncating the remaining faces of Δ_G^* labelled with L_I for tubes I, in increasing order of dimension, results in the pseudograph associahedron P_G. If G is a pseudograph with connected components $G^1, \ldots, G^q$, then the pseudograph associahedron is $P_G = P_{G^1} \times \cdots \times P_{G^q}$.[4]

Example 2.4. Let G be a pseudograph with three nodes and one bundle of size two labelled as in Figure 2. Then the pseudograph G has nine tubes, and five of them are full. The following table shows all the tubes I of G and their labellings L_I.

I	L_I		I	L_I		I	L_I	
1	$1ab$	full	23	$23ab$	full	$12a$	$12a$	not full
2	$2ab$	full	$12ab$	$12ab$	full	$12b$	$12b$	not full
3	$3ab$	full				$123a$	$123a$	not full
						$123b$	$123b$	not full

First, we truncate the faces of Δ_G corresponding to full tubes. Then we get the first polytope in Figure 3 and the defunct tubes $12a$ and $12b$ correspond to some edges of the new facet obtained from the truncation of the face corresponding to $12ab$. Now we truncate the faces of Δ_G^* whose labels correspond to the remaining tubes. Then we obtain the pseudograph associahedron with nine facets labelled by L_I. See Figure 3.

For simplicity, we will use the same notation for tubes and facets, that is, we denote by I the facet labelled by L_I. The face poset of P_G is isomorphic to the set of tubings

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[4] We note that, for a connected simple graph G, we have $P_G = Q_G$, where P_G is the graph associahedron corresponding to G as in (1.3), and Q_G is the polytope defined by the original definition of pseudograph associahedron in [3]. However, if G is not connected, that is, G has connected components $G^1, \ldots, G^q$, then Q_G is defined to be $Q_G = P_{G^1} \times \cdots \times P_{G^q} \times \Delta^{q-1}$ while $P_G = P_{G^1} \times \cdots \times P_{G^q}$ by (1.3). In order to avoid this confusion, we shall use a slightly modified definition of pseudograph associahedron as in Definition 2.3.
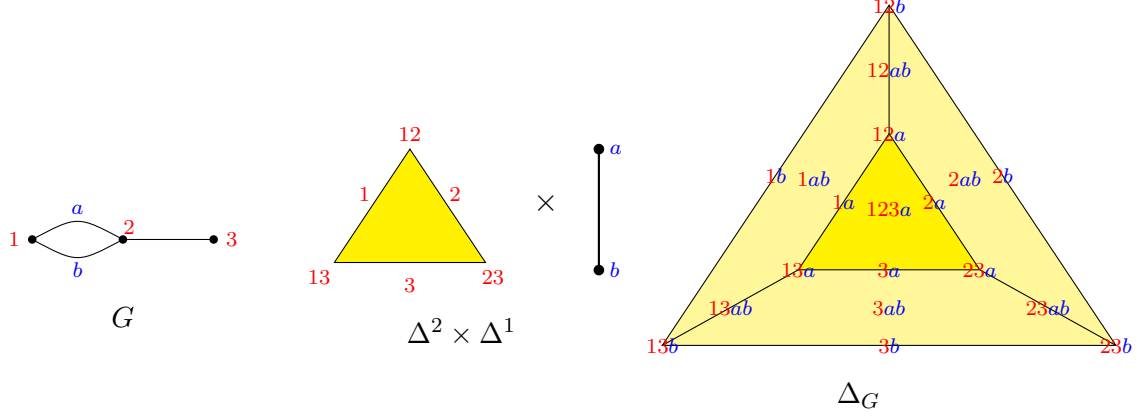
FIGURE 2. An example of Δ_G whose facets are labelled with L_I
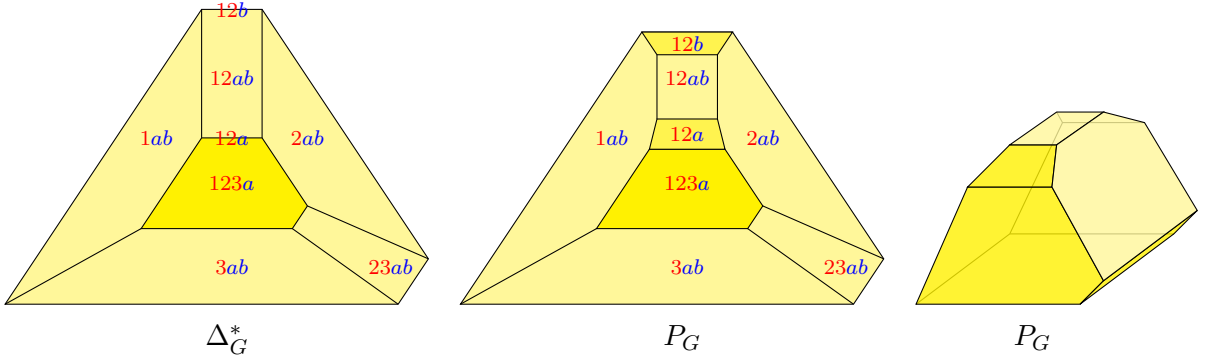
FIGURE 3. The pseudograph associahedron P_G

of G, ordered under reverse subset containment. Two facets *meet by inclusion* (respectively, *meet by separation*) if their corresponding tubes meet by inclusion (respectively, meet by separation). In addition, the set of tubes of G also represents the set of facets of P_G. An n-dimensional convex polytope P is called a *Delzant polytope* if the outward normal vectors to the facets meeting at each vertex form an integral basis of $\mathbb{Z}^n$. In the rest of this section, we shall discuss the Delzant construction of P_G.

Lemma 2.5. *Let P be a Delzant polytope and F a proper face of P. Then there is the canonical truncation of P along F such that the result is a Delzant polytope.*

Proof. Assume $\dim P = n$. Suppose that F is the intersection of facets $F_1, \ldots, F_k$ for $1 \leq k \leq n$ whose outward normal vectors are $\lambda(F_1), \ldots, \lambda(F_k)$, respectively. We truncate P along F by a hyperplane whose normal vector is $\lambda(F_1) + \cdots + \lambda(F_k)$, and obtain a convex simple polytope $\mathrm{cut}_F(P)$. We claim that $\mathrm{cut}_F(P)$ is Delzant. In order to prove the claim, by the convexity of $\mathrm{cut}_F(P)$, the only thing what we have to show is that for every new vertex v of $\mathrm{cut}_F(P)$, the normal vectors of facets containing v

are unimodular. If $k = 1$, then $\mathrm{cut}_F(P)$ is clearly a Delzant polytope. Now assume that $k > 1$. Then at each vertex v of F, there exist facets $F_{k+1}^v, \ldots, F_n^v$ such that the outward normal vectors $\lambda(F_1), \ldots, \lambda(F_k), \lambda(F_{k+1}^v), \ldots, \lambda(F_n^v)$ form an integral basis of $\mathbb{Z}^n$. Let $\tilde{F}$ be the facet of $\mathrm{cut}_F(P)$ which is the new facet arising from the truncation of the face F. Note that the facets $F_1, \cdots, F_k$ and $F_{k+1}^v, \ldots, F_n^v$ are also cut when we truncate the face F of P. Let $\tilde{F}_1, \ldots, \tilde{F}_k, \tilde{F}_{k+1}^v, \ldots, \tilde{F}_n^v$ be the facets of $\mathrm{cut}_F(P)$ which are the cuts of $F_1, \ldots, F_k, F_{k+1}^v, \ldots, F_n^v$, respectively. We note that each new vertex of $\tilde{F}$ can be written as the intersection of $\tilde{F}_1, \ldots, \tilde{F}_{i-1}, \tilde{F}, \tilde{F}_{i+1}, \ldots, \tilde{F}_k, \tilde{F}_{k+1}^v, \ldots, \tilde{F}_n^v$, where v is a vertex of F. Furthermore, in this case, $\{F_1, \ldots, F_k, F_{k+1}^v, \ldots, F_n^v\}$ should meet at the vertex v in P. Since P is Delzant, their normal vectors are unimodular. Therefore, we have

$$\det \begin{pmatrix} \lambda(F_1) & \cdots & \lambda(F_k) & \lambda(F_{k+1}^v) \cdots & \lambda(F_n^v) \end{pmatrix}$$
$$= \det \begin{pmatrix} \lambda(\tilde{F}_1) & \cdots & \lambda(\tilde{F}_{i-1}) & \lambda(\tilde{F}) & \lambda(\tilde{F}_{i+1}) & \cdots & \lambda(\tilde{F}_k) & \lambda(\tilde{F}_{k+1}^v) & \cdots & \lambda(\tilde{F}_n^v) \end{pmatrix},$$

and hence $\mathrm{cut}_F(P)$ is also Delzant. $\qquad\square$

Let G be a connected pseudograph with node set $V = [n+1]$ and bundles $B_1, \ldots, B_k$, and let $|B_i| = b_i + 1$ for $i = 1, \ldots, k$. Each edge in a bundle B_i is labelled by e_i^j for $i \in [k]$ and $j \in [b_i + 1]$. We set $\mathcal{C}_G = V \cup B_1 \cup \cdots \cup B_k$ and $\mathcal{R}_G = \mathcal{C}_G \setminus \{n+1, e_1^{b_1+1}, \ldots, e_k^{b_k+1}\}$. We note that Δ_G is $|\mathcal{R}_G|$-dimensional. Consider an integral matrix $A = (a_{\alpha,\beta})$ of size $|\mathcal{R}_G| \times |\mathcal{C}_G|$ whose rows are labelled by elements in $\mathcal{R}_G$ and columns are labelled by elements in $\mathcal{C}_G$. Then, there is a Delzant realization of Δ_G such that the normal vector of the facet corresponding to $\beta \in \mathcal{C}_G$ is the column A_β of A labelled by β if we put

$$a_{\alpha,\beta} = \begin{cases} -1, & \text{if } \alpha = \beta; \\ 1, & \text{if } \alpha \in V \text{ and } \beta = n+1, \text{ or } \alpha \in B_i \text{ and } \beta = e_i^{b_i+1}; \\ 0, & \text{otherwise.} \end{cases}$$

Since P_G is obtained from Δ_G by truncating facets canonically, it follows from Lemma 2.5 that for a pseudograph G, the pseudograph associahedron P_G becomes a Delzant polytope. More precisely, the outward normal vector of the facet corresponding to a tube I is $\sum_{\beta \in L_I} A_\beta = \sum_{\beta \in I} A_\beta$. Let λ_G be the mod 2 characteristic function of P_G. Then, $\lambda_G(I) \equiv \sum_{\beta \in I} A_\beta \pmod 2$. For simplicity, let $M_G := M_{\lambda_G}(P_G)$, which is a real toric manifold associated with λ_G.

3. THE a-POLYNOMIAL OF A PSEUDOGRAPH AND THE MAIN RESULT

In this section, we will define the a-polynomial of a pseudograph and explain our main result.

Let G be a connected pseudograph with node set V and bundles $B_1, \ldots, B_k$, and $\mathcal{C}_G = V \cup B_1 \cup \cdots \cup B_k$. We set

$$2_{\mathrm{even}}^{\mathcal{C}_G} = \{C \subset \mathcal{C}_G \mid |C \cap V| \equiv |C \cap B_1| \equiv \cdots \equiv |C \cap B_k| \equiv 0 \pmod 2\} \subset 2^{\mathcal{C}_G}.$$

A collection $C \subset \mathcal{C}_G$ is said to be *admissible* to G if it satisfies the following (a1)~(a3):

   (a1) $C \in 2_{\mathrm{even}}^{\mathcal{C}_G}$,

(a2) C contains the nodes which are not endpoints of any bundle of G, and

(a3) for each bundle B of G, $B \cap C \neq \emptyset$.

If G has q connected components $G^1, \ldots, G^q$, then $C \subset \mathcal{C}_G$ is said to be *admissible* to G if $C \cap G^i$ is admissible to G^i for every $i = 1, \ldots, q$. If $C \subset \mathcal{C}_G$ is admissible to G, then C is called an *admissible collection* of G. For example, Figure 4 shows all admissible collections of the pseudograph G in Figure 2 and Figure 5 shows some non-admissible collections.
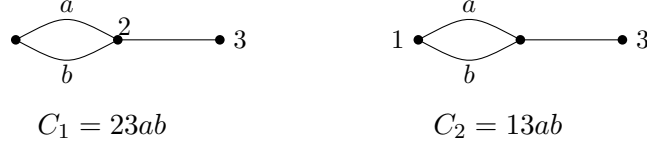

$$C_1 = 23ab \qquad\qquad\qquad C_2 = 13ab$$

FIGURE 4. Admissible collections

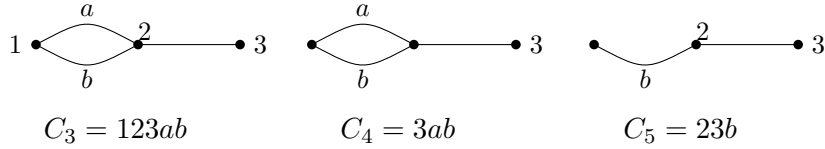$$C_3 = 123ab \qquad\qquad C_4 = 3ab \qquad\qquad C_5 = 23b$$

FIGURE 5. Non-admissible collections

For a pseudograph G and for each collection $C \subset \mathcal{C}_G$, let $P_{C,G}^{\mathrm{odd}}$ denote the union of all facets I of P_G such that $|I \cap C|$ is odd, and $K_{C,G}^{\mathrm{odd}}$ its dual simplicial complex.

For a topological space X, if X has the homotopy type of a finite CW-complex, then we let $\mathrm{Poin}_X(t)$ and $\widetilde{\mathrm{Poin}}_X(t)$ be the polynomials defined by

$$\mathrm{Poin}_X(t) = \sum_{i=0}^{\infty} \beta^i(X)t^i \quad \text{and} \quad \widetilde{\mathrm{Poin}}_X(t) = \sum_{i=0}^{\infty} \tilde{\beta}^i(X)t^i, \text{ respectively,}$$

where $\beta^i(X) = \mathrm{rank}_{\mathbb{Q}} H^i(X; \mathbb{Q})$ and $\tilde{\beta}^i(X) = \mathrm{rank}_{\mathbb{Q}} \widetilde{H}^i(X; \mathbb{Q})$. Then we define the *a-polynomial* $a_G(t)$ of G by

$$(3.1) \qquad\qquad a_G(t) = \sum_{\substack{C \subset \mathcal{C}_G \\ \text{admissible to } G}} \widetilde{\mathrm{Poin}}_{K_{C,G}^{\mathrm{odd}}}(t).$$

Note that if G has no admissible collection then the *a*-polynomial $a_G(t)$ of G is defined to be the zero polynomial. Hence, if G is a simple graph with odd number of nodes, then $a_G(t)$ is the zero polynomial. For example, recall that the pseudograph G in Figure 2 has only two admissible collections C_1 and C_2 in Figure 4. Figure 6 says that $K_{C_1,G}^{\mathrm{odd}}$ is homotopy equivalent to S^1, and $K_{C_2,G}^{\mathrm{odd}}$ is null-homotopic. Thus $\widetilde{\mathrm{Poin}}_{K_{C_1,G}^{\mathrm{odd}}}(t) = t$ and $\widetilde{\mathrm{Poin}}_{K_{C_2,G}^{\mathrm{odd}}}(t) = 0$, and hence the *a*-polynomial of G is $a_G(t) = t$.
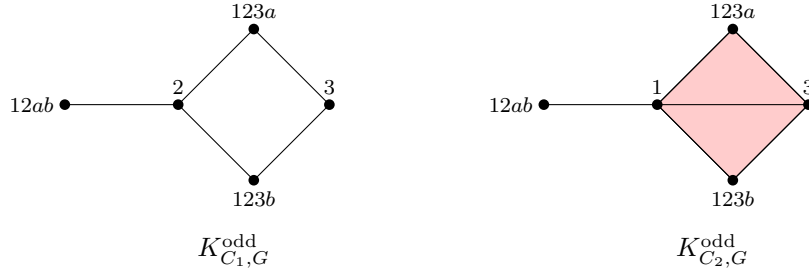
FIGURE 6. Simplicial complexes corresponding to admissible collections

Note that for a pseudograph G, by (1.1), we already know that the rational Betti numbers of the real toric manifold M_G corresponding to G are computed as

$$\mathrm{Poin}_{M_G}(t) = 1 + t \sum_{\omega \in \mathrm{Row}(\lambda_G)} \widetilde{\mathrm{Poin}}_{(P_G)_\omega}(t),$$

where λ_G is the mod 2 characteristic function of M_G defined at the end of Section 2.

We define a matrix $\Lambda' = (\lambda'_{\alpha,I})$ over $\mathbb{Z}_2$ whose rows are labelled by elements in $\mathcal{C}_G$ and columns are labelled by the tubes of G such that

$$\lambda'_{\alpha,I} = \begin{cases} 1 & \text{if } \alpha \in I \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a connected pseudograph with node set $V = [n+1]$ and bundles $B_1, \ldots, B_k$, where $|B_i| = b_i + 1$ for each $i = 1, \ldots, k$. Then λ_G can be obtained from Λ' by adding the row Λ'_{n+1} to the rows $\Lambda'_1, \ldots, \Lambda'_n$, adding the row $\Lambda'_{e_i^{b_i+1}}$ to the rows $\Lambda'_{e_i^1}, \ldots, \Lambda'_{e_i^{b_i}}$ for $i = 1, \ldots, k$, and then deleting the rows $\Lambda'_{n+1}, \Lambda'_{e_1^{b_1+1}}, \ldots, \Lambda'_{e_k^{b_k+1}}$ from Λ', where Λ'_α is the row vector of Λ' corresponding to $\alpha \in \mathcal{C}_G$. Each subset C of $\mathcal{C}_G$ is assigned to an element $\omega_C := \sum_{c \in C} \Lambda'_c$ of $\mathrm{Row}(\Lambda')$. Define

$$\mathrm{Row}(\Lambda'_{\mathrm{even}}) := \{\omega_C \in \mathrm{Row}(\Lambda') \mid C \in 2^{\mathcal{C}_G}_{\mathrm{even}}\}.$$

Then, $\mathrm{Row}(\Lambda'_{\mathrm{even}})$ is a subspace of $\mathrm{Row}(\Lambda')$.

Let $\mathcal{R}_G = \mathcal{C}_G \setminus \{n+1, e_1^{b_1+1}, \ldots, e_k^{b_k+1}\}$. For $R \subset \mathcal{R}_G$, there is a bijection from $2^{\mathcal{R}_G}$ to $2^{\mathcal{C}_G}_{\mathrm{even}}$ defined by $R \mapsto C$, where

$$C \cap V = \begin{cases} R \cap V & \text{if } |R \cap V| \text{ is even;} \\ (R \cap V) \cup \{n+1\} & \text{if } |R \cap V| \text{ is odd,} \end{cases}$$

and for $i = 1, \ldots, k$,

$$C \cap B_i = \begin{cases} R \cap B_i & \text{if } |R \cap B_i| \text{ is even;} \\ (R \cap B_i) \cup \{e_i^{b_i+1}\} & \text{if } |R \cap B_i| \text{ is odd.} \end{cases}$$

Note that each element of $\mathrm{Row}(\lambda_G)$ is associated with a subset R in $\mathcal{R}$, and this bijection identifies $\mathrm{Row}(\lambda_G)$ with $\mathrm{Row}(\Lambda'_{\mathrm{even}})$. Furthermore, one can see that $(P_G)_{\omega_C}$

is the union of all facets I such that $|I \cap C|$ is odd. Hence, by (1.1), the ith rational Betti number $\beta^i(M_G)$ of M_G is given by

$$(3.2) \qquad \beta^i(M_G) = \sum_{C \in 2^{\mathcal{C}_G}_{\text{even}}} \tilde{\beta}^{i-1}(P^{\text{odd}}_{C,G}).$$

Hence, for a pseudograph G,

$$\text{Poin}_{M_G}(t) = 1 + t \sum_{C \in 2^{\mathcal{C}_G}_{\text{even}}} \widetilde{\text{Poin}}_{K^{\text{odd}}_{C,G}}(t).$$

Recall that a pseudograph G' is a partial underlying pseudograph of G if G' can be obtained from G by replacing some bundles with simple edges, that is, the set of all the bundles of G' is a subset of that of G. We denote by $H \lessdot G$ if H is a partial underlying pseudograph of an induced subgraph of G. See Figure 7 for illustrations. We now restate the main theorem. The proof will be presented in Section 4.

Theorem 1.1 *For any pseudograph G, we have*

$$\text{Poin}_{M_G}(t) = 1 + t \sum_{H \lessdot G} a_H(t) = 1 + t \sum_{H \lessdot G} \sum_{\substack{C \subset \mathcal{C}_H \\ \text{admissible to } H}} \widetilde{\text{Poin}}_{K^{\text{odd}}_{C,H}}(t).$$

One should note that if $H \lessdot G$, then we have $\mathcal{C}_H \subseteq \mathcal{C}_G$ and the dimension of $K^{\text{odd}}_{C,H}$ is much less than the dimension of $K^{\text{odd}}_{C,G}$ for each collection $C \subset \mathcal{C}_H$. We will see in the next section that for each collection C admissible to H, $K^{\text{odd}}_{C,H}$ is a simplicial subcomplex of $K^{\text{odd}}_{C,G}$, and $K^{\text{odd}}_{C,H}$ and $K^{\text{odd}}_{C,G}$ have the same homotopy type.

Here is an example. Let us compute $\text{Poin}_{M_G}(t)$ for the pseudograph G in Figure 2. Figure 7 shows all pseudographs $H \lessdot G$, where only four pseudographs can have an admissible collection, that is, $H_1 = 123ab$, $H_2 = 12ab$, $H_3 = 23$, and $H_4 = 12$. We computed that $a_{H_1}(t) = t$ right after the definition of the a-polynomial. The admissible collections of H_2 are $12ab$ and ab, where $K^{\text{odd}}_{12ab,H_2}$ and K^{odd}_{ab,H_2} are homotopy equivalent to S^1 and S^0, respectively. Thus $a_{H_2}(t) = 1 + t$. Since 23 is the only admissible collection of H_3 and K^{odd}_{23,H_3} is homotopy equivalent to S^0, we have $a_{H_3}(t) = 1$. In addition, 12 is the only admissible collection of H_4 and K^{odd}_{12,H_4} is homotopy equivalent to S^0. Hence, $a_{H_4}(t) = 1$. Thus $\text{Poin}_{M_G}(t) = 1 + t(t + (1 + t) + 2) = 1 + 3t + 2t^2$.
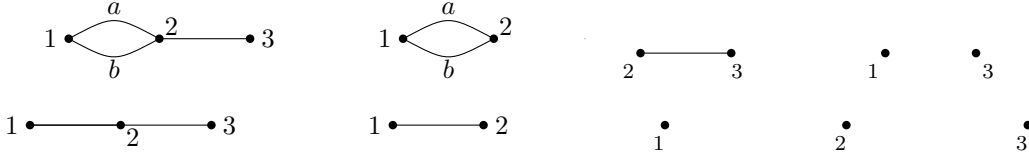

FIGURE 7. All pseudographs $H \lessdot G$ for the pseudograph G in Figure 2

4. A SIMPLICIAL COMPLEX $K_{C,G}^{\text{odd}}$ AND ITS SUBCOMPLEXES

In this section, we will show that a simplicial complex $K_{C,G}^{\text{odd}}$ is homotopy equivalent to $K_{C,H}^{\text{odd}}$ for some $H \lessdot G$, and we will prove Theorem 1.1.

Let G be a connected pseudograph with node set $V = [n+1]$ and bundles $B_1, \ldots, B_k$, where $|B_i| = b_i + 1$ for each $i = 1, ..., k$, as in the previous section.

The following lemma is useful to find a simplicial subcomplex which is homotopy equivalent to a given simplicial complex.

Lemma 4.1 (Lemma 5.2 of [5]). *Let I be a vertex of a simplicial complex K and suppose that the link $\operatorname{Lk} I$ of I in K is contractible. Then K is homotopy equivalent to the complex $K \setminus \operatorname{St} I$, where $\operatorname{St} I$ is the star of I.*

For each $C \in 2_{\text{even}}^{\mathcal{C}_G}$, let $\widetilde{\Gamma}_G(C)$ be the subgraph of G induced by the node set $V_C := \{v \in V \mid v \in C \text{ or } v \text{ is an endpoint of an edge } e \in C\}$. We define a simplicial subcomplex $K'_{C,G}$ of $K_{C,G}^{\text{odd}}$ by

$$\{\text{the vertices of } K'_{C,G}\} = \{\text{tubes } I \text{ of } G \text{ such that } I \subseteq \widetilde{\Gamma}_G(C) \text{ and } |I \cap C| \text{ is odd}\}.$$

We note that the vertices of $K'_{C,G}$ are tubes of $\widetilde{\Gamma}_G(C)$ unless $I = \widetilde{\Gamma}_G(C)$.

Example 4.2. Let G be a connected pseudograph with four nodes and two bundles in Figure 8. Let $C_1 = 13ab$ and $C_2 = 12cd$, then $\widetilde{\Gamma}_G(C_1) = 123ab$ and $\widetilde{\Gamma}_G(C_2) = 124abcd$.
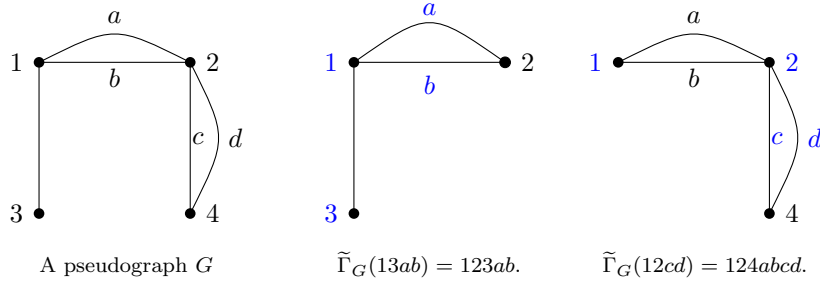

FIGURE 8. Examples of $\widetilde{\Gamma}_G$

Note that the vertices of $K'_{C_1,G}$ are

$$1, 3, 12ab, 123a, \text{ and } 123b,$$

and the vertices of $K_{C_1,G}^{\text{odd}}$ are $124abc, 124abd, 124abcd, 1234ac, 1234ad, 1234acd, 1234bc,$ $1234bd, 1234bcd$, together with vertices of $K'_{C_1,G}$. Figure 9 shows that $K_{C_1,G}^{\text{odd}}$ and $K'_{C_1,G}$ are homotopy equivalent.

The following lemma shows that the phenomenon in the above example holds for any $C \in 2_{\text{even}}^{\mathcal{C}_G}$.

Lemma 4.3. $K_{C,G}^{\text{odd}}$ *is homotopy equivalent to $K'_{C,G}$.*
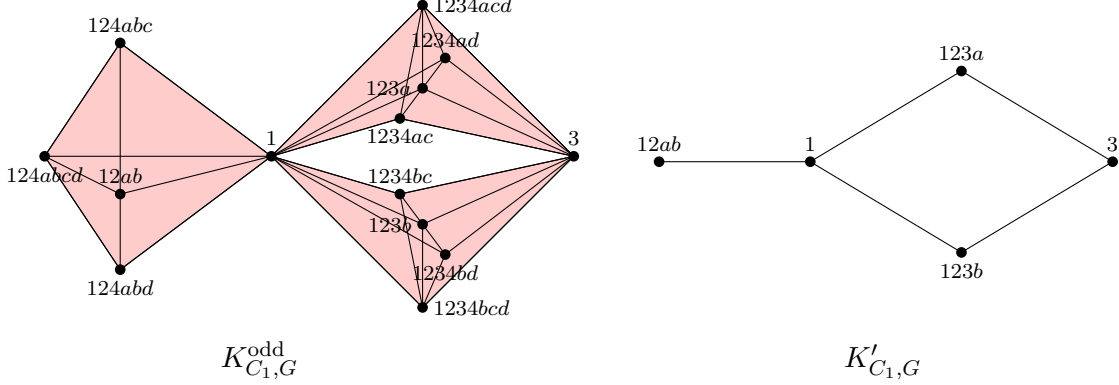
FIGURE 9. For the pseudograph G and $C_1 = 13ab$ in Figure 8, two simplicial complexes $K^{\mathrm{odd}}_{C_1,G}$ and $K'_{C_1,G}$ are homotopy equivalent.

Proof. It is enough to consider the case when $\widetilde{\Gamma}_G(C)$ is a proper subgraph of G. For simplicity, we write K_G and K' instead of $K^{\mathrm{odd}}_{C,G}$ and $K'_{C,G}$, respectively. We will show that we can eliminate the stars of all vertices in $K_G \setminus K'$, one by one, from K_G to K', without changing the homotopy type.

Suppose that we cannot eliminate the stars of all vertices in $K_G \setminus K'$ one by one, without changing the homotopy type. Then we can obtain a minimal complex $K^* \supsetneq K'$, which is obtained by eliminating the stars of some vertices in $K_G \setminus K'$ without changing the homotopy type. Then take a vertex I which is minimal in $K^* \setminus K'$. Since $I \in K_G$ and $I \notin K'$, $|I \cap C|$ is odd and $I \setminus \widetilde{\Gamma}_G(C) \neq \emptyset$. Since $\widetilde{\Gamma}_G(C)$ is an induced subgraph, $I \setminus \widetilde{\Gamma}_G(C)$ contains a node. Let $\widetilde{I}$ be the subgraph of I which is obtained from I by deleting the nodes not in $\widetilde{\Gamma}_G(C)$. Note that from the definitions of $\widetilde{\Gamma}_G(C)$ and $\widetilde{I}$, $I \cap C = \widetilde{I} \cap C$. Then we can choose a connected component $\widetilde{I}_0$ of $\widetilde{I}$ such that $|\widetilde{I}_0 \cap C|$ is odd. Since $\widetilde{I}_0 \subset \widetilde{I} \subset \widetilde{\Gamma}_G(C)$, we conclude that $\widetilde{I}_0 \in K'$.

We will show that any vertex of $\mathrm{Lk}\, I$ meets with $\widetilde{I}_0$. Let $L \in \mathrm{Lk}\, I$ in K^*. If I and L meet by separation, then $\widetilde{I}_0$ and L also meet by separation. Now assume that I and L meet by inclusion. If $I \subset L$, then $\widetilde{I}_0 \subset L$. Suppose that $L \subsetneq I$. The minimality of I implies that $L \in K'$, and hence $L \subseteq \widetilde{\Gamma}_G(C)$. As L is a connected subgraph contained in I and $I \cap \widetilde{\Gamma}_G(C) = \widetilde{I}$, L must be contained in some connected component of $\widetilde{I}$. Hence, either L is contained in $\widetilde{I}_0$, or L and $\widetilde{I}_0$ meet by separation. Therefore, $\mathrm{Lk}\, I$ is contractible and K^* is homotopy equivalent to $K^* \setminus \mathrm{St}\, I$ by Lemma 4.1. Then $K^* \setminus \mathrm{St}\, I$ is smaller than K^*, which contradicts the minimality of K^*. □

Let us consider a simplicial subcomplex $K''_{C,G}$ of $K'_{C,G}$ whose vertex set consists of vertices I of $K'_{C,G}$ such that, for each bundle B of $\widetilde{\Gamma}_G(C)$ satisfying $B \cap C = \emptyset$, if the endpoints of B are in I, then $B \subset I$. That is, we obtain $K''_{C,G}$ from $K'_{C,G}$ by removing the stars of the vertices I if there exists a bundle B of $\widetilde{\Gamma}_G(C)$ such that $B \cap C = \emptyset$, I contains the endpoints of B, and $B \not\subset I$.

Back to Example 4.2, the vertices of $K''_{C_2,G}$ are

$$1, 2, 24cd, 124abc, \text{ and } 124abd,$$

and the vertices of $K'_{C_2,G}$ are $124ac, 124bc, 124ad, 124bd$, together with the vertices of $K''_{C_2,G}$. Figure 10 shows that $K'_{C_2,G}$ and $K''_{C_2,G}$ are homotopy equivalent.
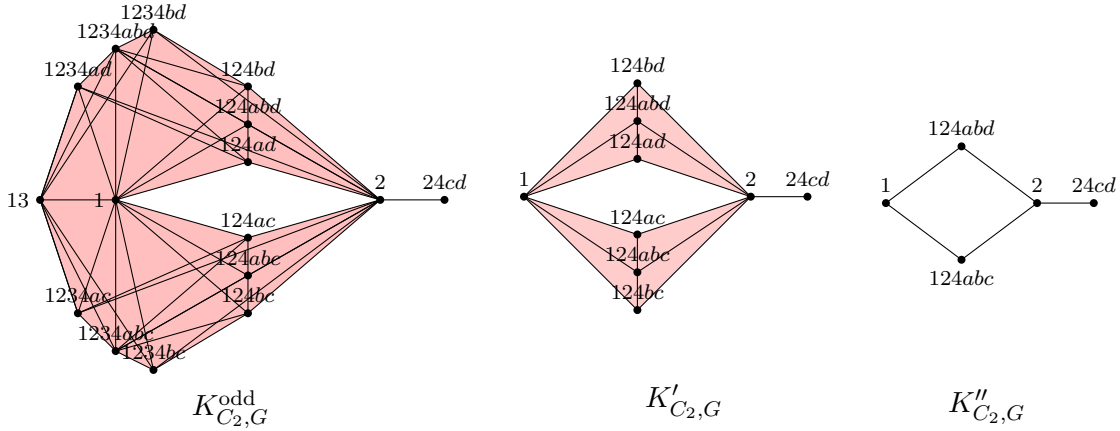

FIGURE 10. For $C_2 = 12cd$, three simplicial complexes $K^{\text{odd}}_{C_2,G}$, $K'_{C_2,G}$, and $K''_{C_2,G}$ are homotopy equivalent.

Lemma 4.4. $K'_{C,G}$ is homotopy equivalent to $K''_{C,G}$.

Proof. For simplicity, we write K' and K'' instead of $K'_{C,G}$ and $K''_{C,G}$, respectively. We give a surjective map from the vertices of K' to the vertices of K'' so that $I \in K'$ corresponds to $\widetilde{I} \in K''$ where $\widetilde{I}$ is a subgraph of G obtained from I by transforming $I \cap B$ (if it is non-empty) into a bundle B whenever there exists a bundle B of G such that $C \cap B = \emptyset$. Note that for any $I \in K'$, $I = \widetilde{I}$ if and only if $I \in K''$.

Now we will show that we can eliminate the stars of all vertices in $K' \backslash K''$, one by one, from K' to K'', without changing the homotopy type. Suppose we cannot eliminate the stars of all vertices in $K' \setminus K''$ one by one, without changing the homotopy type. Then we can obtain a minimal complex $K^* \supsetneq K''$, which is obtained by eliminating the stars of some vertices in $K' \setminus K''$ as long as eliminating does not change the homotopy type. Then take a vertex I which is maximal in $K^* \setminus K''$. Then $I \subsetneq \widetilde{I}$ and $\widetilde{I} \in K''$. We will show that $\widetilde{I}$ meets every vertex in $\operatorname{Lk} I$. Take $L \in \operatorname{Lk} I$. If I and L meet by separation, then so do $\widetilde{I}$ and L because I and $\widetilde{I}$ have the same node set. If $L \subseteq I$, then clearly $L \subseteq \widetilde{I}$. Now assume that $I \subsetneq L$. Then the maximality of I implies $L \in K''$. Therefore $L = \widetilde{L}$. Since $I \subsetneq L$, $\widetilde{I} \subset \widetilde{L} = L$. Thus $\widetilde{I}$ and L meet by inclusion. Therefore K^* is homotopy equivalent to $K^* \setminus \operatorname{St}(I)$ by Lemma 4.1. Then $K^* \setminus \operatorname{St} I$ is smaller than K^*, which contradicts the minimality of K^*. $\square$

By Lemmas 4.3 and 4.4, formula (3.2) for the ith rational Betti number of M_G for a connected pseudograph G can be written as follows:

$$(4.1) \qquad \beta^i(M_G) = \sum_{C \in 2^{\mathcal{C}_G}_{\text{even}}} \tilde{\beta}^{i-1}(K''_{C,G}).$$

Before we give a proof of Theorem 1.1, let us prepare the following lemma which enables us to determine whether $K''^{\text{odd}}_{C,G}$ is contractible or not for a given collection $C \in 2^{\mathcal{C}_G}_{\text{even}}$. For $C \in 2^{\mathcal{C}_G}$, a subgraph J of a pseudograph G is *odd* (respectively, *even*) with respect to C if $|J \cap C|$ is odd (respectively, even).

Lemma 4.5. *For $C \in 2^{\mathcal{C}_G}_{\text{even}}$, let $\Gamma^1, \ldots, \Gamma^q$ be the connected components of $\widetilde{\Gamma}_G(C)$, and let $C^j = C \cap \Gamma^j$ for $j = 1, \ldots, q$.*

   (i) *If some connected component of $\widetilde{\Gamma}_G(C)$ is odd with respect to C (equivalently, $|C^j|$ is odd for some j), then $K''_{C,G}$ is contractible.*

   (ii) *The simplicial complex $K''_{C,G}$ is the simplicial join of all $K''_{C^j,G}$'s for $j = 1, \ldots, q$, and hence*

$$\widetilde{H}_{i-1}(K''_{C,G}; \mathbb{Q}) = \bigoplus_{\sum k_j = i-q} \bigotimes \widetilde{H}_{k_j}(K''_{C^j,G}; \mathbb{Q}).$$

Proof. We first assume that some connected component of $\widetilde{\Gamma}_G(C)$ is odd with respect to C. Without loss of generality, we may assume that I^1 is odd with respect to C. Then I^1 is a vertex of $K''_{C,G}$. Let J be a tube in $K''_{C,G}$. Then J is contained in Γ^j for some j. If $j = 1$, then $J \subset \Gamma^1$, and hence Γ^1 and J meet by inclusion. If $j \neq 1$ then Γ^1 and J meet by separation. Therefore Γ^1 meets every other vertex of $K''_{C,G}$. Hence, $K''_{C,G}$ is contractible, which proves the statement (i).

Note that $\Gamma^j = \widetilde{\Gamma}_G(C^j)$ for $j = 1, \ldots, q$. If $q = 1$ then $C = C^1$ and so (ii) is true. Suppose that $q \geq 2$. Note that each vertex I of $K''_{C,G}$ corresponds to a vertex of $K''_{C^i,G}$ for some i. Take two vertices I and J of $K''_{C,G}$ such that $I \subset \Gamma^i$ and $J \subset \Gamma^j$ for $i \neq j$. Then, I and J meet by separation, and so the simplex spanned by I and J in $K''_{C,G}$ corresponds to the simplex in $K''_{C^i,G} * K''_{C^j,G}$. Note that the join $A * B$ is homotopy equivalent to the (reduced) suspension of the smash product $A \wedge B$, and $\Sigma(A \wedge B) = S^1 \wedge A \wedge B$. Hence, we have

$$\begin{aligned} K''_{C,G} &= K''_{C^1,G} * \cdots * K''_{C^q,G} \\ &\simeq \underbrace{S^1 \wedge \cdots \wedge S^1}_{q-1} \wedge K''_{C^1,G} \wedge \cdots \wedge K''_{C^q,G} \\ &= S^{q-1} \wedge K''_{C^1,G} \wedge \cdots \wedge K''_{C^q,G}. \end{aligned}$$

Therefore, we have

$$\widetilde{H}_{i-1}(K''_{C,G}; \mathbb{Q}) = \bigoplus_{\sum k_j = i-q} \bigotimes \widetilde{H}_{k_j}(K''_{C^j,G}; \mathbb{Q})$$

as desired in (ii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $2^{\mathcal{C}_G}_{\text{even}*}$ be the set of collections C in $2^{\mathcal{C}_G}_{\text{even}}$ such that the intersection of C and each of the connected components of $\widetilde{\Gamma}_G(C)$ belongs to $2^{\mathcal{C}_G}_{\text{even}}$, that is,

$$2^{\mathcal{C}_G}_{\text{even}*} = \{C \in 2^{\mathcal{C}_G}_{\text{even}} \mid \text{each connected component of } \widetilde{\Gamma}_G(C) \text{ is even with respect to } C\}.$$

Due to Lemma 4.5, equation (4.1) is equivalent to

$$(4.2) \qquad \beta^i(M_G) = \sum_{C \in 2^{\mathcal{C}_G}_{\text{even}*}} \tilde{\beta}^{i-1}(K''_{C,G}).$$

Now, we obtain a pseudograph $\Gamma_G(C)$ from $\widetilde{\Gamma}_G(C)$ by replacing each bundle B of $\widetilde{\Gamma}_G(C)$ satisfying $C \cap B = \emptyset$ by a (unlabelled) simple edge. Note that $\Gamma_G(C)$ is a partial underlying pseudograph of the induced subgraph $\widetilde{\Gamma}_G(C)$ of G, and hence $\Gamma_G(C) \prec G$.

Example 4.6. Recall the pseudograph G in Figure 8. Then $\Gamma_G(13ab) = 123ab$ and $\Gamma_G(12cd) = 124cd$. See Figure 11.
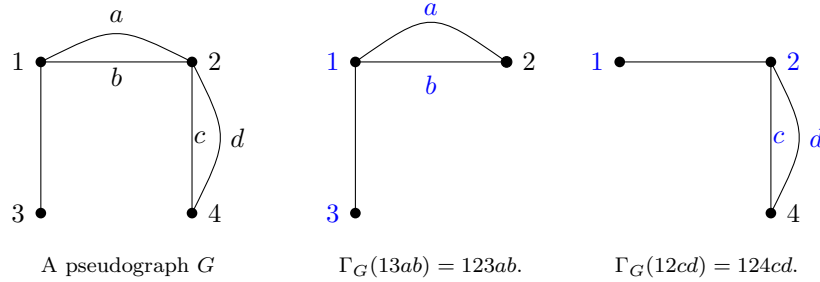

$$\text{A pseudograph } G \qquad \Gamma_G(13ab) = 123ab. \qquad \Gamma_G(12cd) = 124cd.$$

FIGURE 11. Examples of Γ_G

Proposition 4.7. *For $C \in 2^{\mathcal{C}_G}_{\text{even}*}$, $K^{\text{odd}}_{C,G}$ is homotopy equivalent to $K^{\text{odd}}_{C,\Gamma_G(C)}$, and hence, the ith rational Betti number of M_G is*

$$(4.3) \qquad \beta^i(M_G) = \sum_{C \in 2^{\mathcal{C}_G}_{\text{even}*}} \tilde{\beta}^{i-1}(K^{\text{odd}}_{C,\Gamma_G(C)}).$$

Proof. By equation (4.2), it is sufficient to show that $K''_{C,G}$ and $K^{\text{odd}}_{C,\Gamma_G(C)}$ are the same for $C \in 2^{\mathcal{C}_G}_{\text{even}*}$. Figure 12 shows a diagram of this proof.

FIGURE 12. A diagram for the proof of Proposition 4.7, where $C \in 2^{\mathcal{C}_G}_{\text{even}*}$

We first claim that $K'_{C,G}$ is equal to $K^{\mathrm{odd}}_{C,\widetilde{\Gamma}_G(C)}$ for $C \in 2^{\mathcal{C}_G}_{\mathrm{even}*}$. Recall that the vertices of $K'_{C,G}$ are tubes I of G such that I is odd with respect to C and $I \subseteq \widetilde{\Gamma}_G(C)$. By the connectedness of a tube, if I is a vertex of $K'_{C,G}$, then I is contained in some connected component of $\widetilde{\Gamma}_G(C)$ and I is odd with respect to C. If $C \in 2^{\mathcal{C}_G}_{\mathrm{even}*}$, each connected component of $\widetilde{\Gamma}_G(C)$ is even with respect to C by definition. Hence, I is properly contained in some connected component of $\widetilde{\Gamma}_G(C)$, which implies that I is also a tube of $\widetilde{\Gamma}_G(C)$ such that I is odd with respect to C. Therefore, I is also a vertex of $K^{\mathrm{odd}}_{C,\widetilde{\Gamma}_G(C)}$, which implies that $K'_{C,G}$ is a subcomplex of $K^{\mathrm{odd}}_{C,\widetilde{\Gamma}_G(C)}$. Since $\widetilde{\Gamma}_G(C)$ is an induced subgraph of G, if I is a tube of $\widetilde{\Gamma}_G(C)$ such that I is odd with respect to C, then it is also a tube of G such that $I \subsetneq \widetilde{\Gamma}_G(C)$ and I is odd with respect to C. Therefore, every vertex of $K^{\mathrm{odd}}_{C,\widetilde{\Gamma}_G(C)}$ is also a vertex of $K'_{C,G}$, which implies that $K^{\mathrm{odd}}_{C,\widetilde{\Gamma}_G(C)}$ is also a subcomplex of $K'_{C,G}$. This proves the claim.

Note that we obtain $K''_{C,G}$ (respectively $K''_{C,\widetilde{\Gamma}_G(C)}$) from $K'_{C,G}$ (respectively, $K'_{C,\widetilde{\Gamma}_G(C)}$) by removing the stars of the vertices I if there exists a bundle B of $\widetilde{\Gamma}_G(C)$ such that $B \cap C = \emptyset$, I contains the endpoints of B, and $B \not\subset I$. Since $K'_{C,G} = K^{\mathrm{odd}}_{C,\widetilde{\Gamma}_G(C)} = K'_{C,\widetilde{\Gamma}_G(C)}$ for $C \in 2^{\mathcal{C}_G}_{\mathrm{even}*}$, we can see that $K''_{C,G} = K''_{C,\widetilde{\Gamma}_G(C)}$.

From a tube I of $\Gamma_G(C)$, we can find a tube $\widetilde{I}$ of $\widetilde{\Gamma}_G(C)$ such that I is a partial underlying pseudograph of $\widetilde{I}$ as follows; for each bundle B of $\widetilde{\Gamma}_G(C)$ satisfying $B \cap C = \emptyset$, whenever I has an unlabelled simple edge e whose endpoints are the same with the endpoints of B, we transform such a simple edge e into a bundle B. Then $|I \cap C| \equiv |\widetilde{I} \cap C|$ (mod 2), and hence if I is a vertex of $K^{\mathrm{odd}}_{C,\Gamma_G(C)}$, then $\widetilde{I}$ is also a vertex of $K''_{C,\widetilde{\Gamma}_G(C)}$. Now we define a map f from the vertex set of $K^{\mathrm{odd}}_{C,\Gamma_G(C)}$ to the vertex set of $K''_{C,\widetilde{\Gamma}_G(C)}$ by $I \mapsto \widetilde{I}$. Let us show that f is bijective and preserves faces.

(i) f is one-to-one. If I and I' are different vertices of $K^{\mathrm{odd}}_{C,\Gamma_G(C)}$, then they have different node sets or different labelled edges. From the definition of the map f, $\widetilde{I}$ and $\widetilde{I'}$ also have different node sets or different labelled edges. Hence, $f(I)$ and $f(I')$ are different vertices of $K''_{C,\widetilde{\Gamma}_G(C)}$.

(ii) f is surjective. Let J be a vertex of $K''_{C,\widetilde{\Gamma}_G(C)}$. If J does not have a bundle B of $\widetilde{\Gamma}_G(C)$ satisfying $B \cap C = \emptyset$, then J is also an odd tube of $\Gamma_G(C)$ with respect to C, which implies that $f(J) = J$. If J has an bundle B of $\widetilde{\Gamma}_G(C)$ satisfying $B \cap C = \emptyset$, then we can obtain an odd tube I of $\Gamma_G(C)$ with respect to C from J by transforming such bundles B into unlabelled simple edges. Furthermore, $\widetilde{I} = J$, that is, $f(I) = J$.

(iii) f preserves faces. Take two vertices I and I' in $K^{\mathrm{odd}}_{C,\Gamma_G(C)}$. Since I and $\widetilde{I}$ have the same node set, if I and I' meet by separation, then $\widetilde{I}$ and $\widetilde{I'}$ also meet by separation clearly. Now assume that I and I' meet by inclusion. Without loss of generality, we may further assume that $I \subseteq I'$. Hence, if I has an unlabelled simple edge e whose endpoints are the same with the endpoints of B for some bundle B of $\widetilde{\Gamma}_G(C)$ satisfying

$B \cap C = \emptyset$, then I' also has the edge e. Hence, $\widetilde{I} \subseteq \widetilde{I'}$. Thus, if the vertices $I_1, \ldots, I_k$ form a tubing in $\Gamma_G(C)$, then $\widetilde{I}_1, \ldots, \widetilde{I}_k$ also form a tubing in $\widetilde{\Gamma}_G(C)$. Consequently, if the vertices $I_1, \ldots, I_k$ form a simplex in $K^{\mathrm{odd}}_{C, \Gamma_G(C)}$, then $f(I_1), \ldots, f(I_k)$ also form a simplex in $K''_{C, \widetilde{\Gamma}_G(C)}$.

Therefore, $K^{\mathrm{odd}}_{C, \Gamma_G(C)}$ is isomorphic to $K''_{C, \widetilde{\Gamma}_G(C)}$.

$\square$

Remark 4.8. The simplicial subcomplex $K^{\mathrm{odd}}_{C, \Gamma_G(C)}$ is an optimal simplicial subcomplex of $K^{\mathrm{odd}}_{C,G}$ having the same homotopy type, in a sense that $K^{\mathrm{odd}}_{C,H}$ might not have the same homotopy type with $K^{\mathrm{odd}}_{C,G}$ when H is a proper subgraph of $\Gamma_G(C)$. Let G be a complete graph with node set $\{1, 2, 3, 4\}$ and one bundle $B = \{a, b, c\}$, where the endpoints of B are 1 and 2. Then $\Gamma_G(1234ab) = G$. Let us consider a semi-induced subgraph H of G which has four nodes $\{1, 2, 3, 4\}$ and a bundle $B' = \{a, b\}$. Then the simplicial complex $K^{\mathrm{odd}}_{C,H}$ is a simplicial subcomplex of $K^{\mathrm{odd}}_{C,G}$. The simplicial complex $K^{\mathrm{odd}}_{C,G}$ (respectively, $K^{\mathrm{odd}}_{C,H}$) is the order complex of a poset consisting of odd tubes of G (respectively, H) with respect to C, ordered by inclusion. We can compute the reduced Euler characteristic of the order complex of a poset by computing the Möbius function of the poset. Then one can see that the reduced Euler characteristics of $K^{\mathrm{odd}}_{C,G}$ and $K^{\mathrm{odd}}_{C,H}$ are 5 and 1, respectively. Hence, $K^{\mathrm{odd}}_{C,G}$ is not homotopy equivalent to $K^{\mathrm{odd}}_{C,H}$.

Recall that a collection $C \subset \mathcal{C}_G$ is admissible to a connected pseudograph G if it satisfies the following (a1)~(a3):

(a1) $C \in 2^{\mathcal{C}_G}_{\mathrm{even}}$,
(a2) C contains the nodes which are not endpoints of any bundle of G, and
(a3) for each bundle B of G, $B \cap C \neq \emptyset$.

If G has q connected components $G^1, \ldots, G^q$, then $C \subset \mathcal{C}_G$ is admissible to G if $C \cap G^i$ is admissible to G^i for $i = 1, \ldots, q$.

Lemma 4.9. *A collection C belongs to $2^{\mathcal{C}_G}_{\mathrm{even}*}$ if and only if C is admissible to $\Gamma_G(C)$.*

Proof. Let us show the forward direction. Assume that $C \in 2^{\mathcal{C}_G}_{\mathrm{even}*}$. Put $H = \Gamma_G(C)$, and $H^1, \ldots, H^q$ are the connected components of H. Let $C^i = C \cap H^i$ for each $i = 1, \ldots, q$. Then $\Gamma_G(C^i) = H^i$. Hence, it is enough to show that each C^i is admissible to H^i. By the definition of $2^{\mathcal{C}_G}_{\mathrm{even}*}$, each C^i belongs to $2^{\mathcal{C}_{H^i}}_{\mathrm{even}}$, and so (a1) holds. If v is a node in $H^i \setminus C$, then by definition of $\widetilde{\Gamma}_G(C)$, v is an endpoint of some bundle of G, and so (a2) holds. For any bundle B of H^i, by definitions of $\widetilde{\Gamma}_G(C)$ and $\Gamma_G(C)$, $B \cap C \neq \emptyset$, and so (a3) holds. Hence, C^i is admissible to H^i as desired.

The backward direction is clear from (a1) in the definition of admissibility. $\square$

By the above lemma, for $C \in 2^{\mathcal{C}_G}_{\mathrm{even}*}$, $\Gamma_G(C)$ has an admissible collection C. The lemma below says that the converse also holds.

Lemma 4.10. *If $H \lessdot G$ and a collection C is admissible to H, then $H = \Gamma_G(C)$.*

Proof. For a node v of H, by (a2) and (a3), either $v \in C$ or v is an endpoint of some bundle B of H such that $B \cap C \neq \emptyset$, which is exactly equal to the node set of $\Gamma_G(C)$. In addition, by (a3) and the definition of a partial underlying pseudograph, the bundles of H are the bundles of G having a nonempty intersection with C, which exactly coincide with the bundles of $\Gamma_G(C)$. Therefore, $H = \Gamma_G(C)$. $\qquad\square$

Now we are ready to give a proof of our main result, Theorem 1.1.

Proof of Theorem 1.1. For a connected pseudograph G, equation (4.3) in Proposition 4.7 implies

$$\text{Poin}_{M_G}(t) \;=\; 1 + t \sum_{C \in 2^{\mathcal{C}_G}_{\text{even}*}} \widetilde{\text{Poin}}_{K^{\text{odd}}_{C,\Gamma_G(C)}}(t).$$

By Lemmas 4.9 and 4.10,

$$
\begin{aligned}
2^{\mathcal{C}_G}_{\text{even}*} &= \{C \subset \mathcal{C}_G \mid C \text{ is admissible to } \Gamma_G(C)\} \\
&= \dot\cup_{H \lessdot G} \{C \subset \mathcal{C}_H \mid C \text{ is admissible to } H\},
\end{aligned}
$$

where $\dot\cup$ means the disjoint union. Thus we have

$$\text{Poin}_{M_G}(t) = 1 + t \sum_{H \lessdot G} \sum_{\substack{C \subset \mathcal{C}_H \\ \text{admissible to } H}} \widetilde{\text{Poin}}_{K^{\text{odd}}_{C,H}}(t) = 1 + t \sum_{H \lessdot G} a_H(t).$$

If G has q connected components G^1, ..., G^q, then $P_G = P_{G^1} \times \cdots \times P_{G^q}$ and hence $M_G = M_{G^1} \times \cdots \times M_{G^q}$. Therefore,

$$\text{Poin}_{M_G}(t) \;=\; \prod_{i=1}^q \text{Poin}_{M_{G_i}}(t) = \prod_{i=1}^q \left(1 + t \sum_{H^i \lessdot G^i} a_{H^i}(t)\right) = 1 + t^q \sum_{H^1 \lessdot G^1, \dots, H^q \lessdot G^q} \prod_{i=1}^q a_{H^i}(t).$$

By Lemma 4.5,

$$\text{Poin}_{M_G}(t) = 1 + t \sum_{H^1 \lessdot G^1, \dots, H^q \lessdot G^q} a_{H^1 \sqcup \cdots \sqcup H^q}(t) = 1 + t \sum_{H \lessdot G} a_H(t).$$

$\qquad\square$

5. Further remarks on shellability

So far, we have discussed how to compute the rational Betti numbers of the real toric manifold corresponding to a pseudograph associahedron. The amount of the computation of equation (1.1) can be reduced for the pseudograph associahedron associated with a pseudograph G, since we showed that it is sufficient to consider $K^{\text{odd}}_{C,H}$ for only admissible collections $C \in 2^{\mathcal{C}_G}_{\text{even}*}$ of the pseudographs $H \lessdot G$, instead of considering $P^{\text{odd}}_{C,G}$ for all $C \in 2^{\mathcal{C}_G}_{\text{even}}$. Even though our results help to recognize the homotopy type of $P^{\text{odd}}_{C,G}$ by using $K^{\text{odd}}_{C,H}$ for some special pseudograph H, however, we do not have the rich information of the homotopy type of $K^{\text{odd}}_{C,H}$ in general.

We note that it is shown in [5] that if G is a simple graph then for any $C \in 2^{\mathcal{C}_G}_{\text{even}}$, $K^{\text{odd}}_{C,G}$ and its complement $K^{\text{even}}_{C,G}$ are homotopy equivalent to the order complexes of some

posets, respectively, and then they showed that $K_{C,G}^{\mathrm{odd}}$ is homotopy equivalent to a wedge of spheres of the same dimension by showing that the corresponding order complex of $K_{C,G}^{\mathrm{even}}$ is pure and shellable. Therefore it is natural to ask if similar phenomena occur for pseudographs. The former part, the order complex, is naturally extended to pseudographs, but the latter part, the shellability, is not.

Order complex. Let G be a connected pseudograph, and let a collection $C \subset \mathcal{C}_G$ be admissible to G. We will see that $K_{C,G}^{\mathrm{odd}}$ is homotopy equivalent to the order complex of the poset $S_{C,G}^{\mathrm{odd}}$ defined in the following. Let $S_{C,G}^{\mathrm{odd}}$ (respectively, $S_{C,G}^{\mathrm{even}}$) be the set of all subgraphs I of G such that each connected component of I is an odd (respectively, even) tube with respect to C, excluding both $\emptyset$ and C. For example, consider an admissible collection C_2 in Figure 4. Then the vertices of $K_{C_2,G}^{\mathrm{odd}}$ are $1, 3, 12ab, 123a, 123b$ and the elements of $S_{C_2,G}^{\mathrm{odd}}$ are $1, 3, 12ab, 123a, 123b, 13$. Figure 13 shows the order complex of $S_{C_2,G}^{\mathrm{odd}}$.
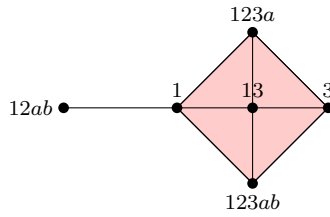

FIGURE 13. The order complex of $S_{C_2,G}^{\mathrm{odd}}$ for an admissible collection $C_2 = 13ab$

Then we can observe that when C is an admissible collection of G, the order complex of $S_{C,G}^{\mathrm{odd}}$ is a geometric subdivision of $K_{C,G}^{\mathrm{odd}}$, because the following are true for a vertex I of the order complex of $S_{C,G}^{\mathrm{odd}}$.

(i) If I is connected, then I is also a vertex of $K_{C,G}^{\mathrm{odd}}$.
(ii) If I consists of connected components $I^1, \ldots, I^\ell$, then $I^1, \ldots, I^\ell$ are vertices of $K_{C,G}^{\mathrm{odd}}$ and they meet by separation.

Since a simplicial complex and its geometric subdivision are homotopy equivalent, $K_{C,G}^{\mathrm{odd}}$ and the order complex of $S_{C,G}^{\mathrm{odd}}$ are homotopy equivalent. Hence, it follows that

$$a_G(t) = \sum_{\substack{C \subset \mathcal{C}_G \\ \text{admissible to } G}} \mathrm{Poin}_{K_{C,G}^{\mathrm{odd}}}(t) = \sum_{\substack{C \subset \mathcal{C}_G \\ \text{admissible to } G}} \mathrm{Poin}_{S_{C,G}^{\mathrm{odd}}}(t).$$

We note that this gives another definition of the a-polynomial, which might be more useful to compute the a-polynomial when $S_{C,G}^{\mathrm{odd}}$'s are shellable.

The order complex of $S_{C,G}^{\mathrm{even}}$ is a geometric subdivision of $K_{C,G}^{\mathrm{even}}$ if we define $K_{C,G}^{\mathrm{even}}$ by the dual simplicial complex of the union of facets corresponding to the tubes of G that are even with respect to C. Hence, since the rational Betti numbers of $K_{C,G}^{\mathrm{odd}}$ are obtained by computing the rational Betti numbers of $K_{C,G}^{\mathrm{even}}$ by Alexander duality, one might obtain $a_G(t)$ by computing $\mathrm{Poin}_{K_{C,G}^{\mathrm{even}}}(t)$ or $\mathrm{Poin}_{S_{C,G}^{\mathrm{even}}}(t)$.

Shellability. A simplicial complex K is *shellable* if its facets can be arranged in linear order $F_1, \ldots, F_t$ in such a way that the subcomplex $(\sum_{i=1}^{k-1} \overline{F_i}) \cap \overline{F_k}$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, t$. For details, see [1].

As we mentioned above, it is shown in [5] that if G is a simple graph then for any $C \in 2^{\mathcal{C}_G}_{\text{even}}$, the order complex $S^{\text{even}}_{C,\Gamma_G(C)}$ is pure and shellable. This cannot be generalized to pseudographs. Actually, there are infinitely many pseudographs such that $S^{\text{odd}}_{C,\Gamma_G(C)}$ is not shellable for some $C \in 2^{\mathcal{C}_G}_{\text{even}}$.
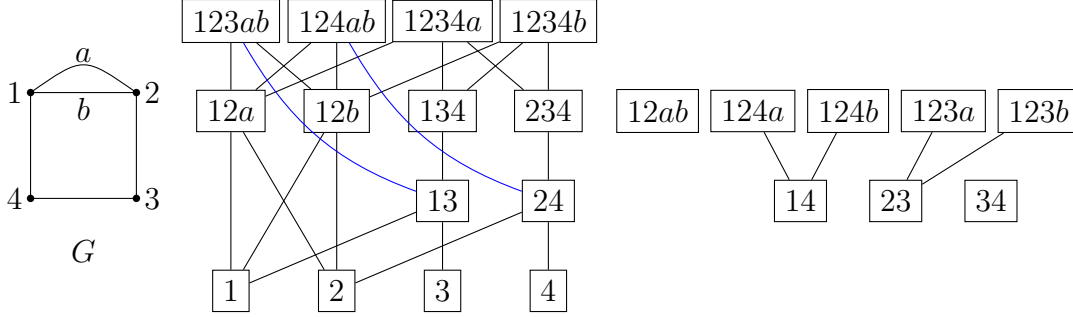

FIGURE 14. Neither $S^{\text{odd}}_{\mathcal{C}_G,G}$ nor $S^{\text{even}}_{\mathcal{C}_G,G}$ is shellable.

Consider the pseudograph in Figure 14. Then, for $C = \mathcal{C}_G$, $\Gamma_G(C) = G$ and neither $S^{\text{odd}}_{C,G}$ nor $S^{\text{even}}_{C,G}$ is shellable. However, both $S^{\text{odd}}_{C,G}$ and $S^{\text{even}}_{C,G}$ are homotopy equivalent to a wedge of spheres; $S^{\text{odd}}_{C,G} \simeq S^2 \vee S^2 \vee S^2$ and $S^{\text{even}}_{C,G} \simeq S^0 \vee S^0 \vee S^0$. Therefore, as a further direction of research, it would be interesting to see whether $S^{\text{odd}}_{C,\Gamma_G(C)}$ is homotopy equivalent to a wedge of spheres for any $C \in 2^{\mathcal{C}_G}_{\text{even}}$ or not. At this moment, all the examples we have are homotopy equivalent to a wedge of spheres. It would also be interesting to characterize pseudographs G such that $S^{\text{even}}_{C,\Gamma_G(C)}$ or $S^{\text{odd}}_{C,\Gamma_G(C)}$ is shellable for any $C \in 2^{\mathcal{C}_G}_{\text{even}*}$.

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